DEPARTMENT OF MATHEMATICS
NEHU, SHILLONG
M.Phil./Ph.D. Model Question

Marks : 100
Time : Two hours

Attempt any 5 questions from Unit I and any 6 from Unit II.

Unit I

(Each question of this unit carries 8 marks.)

State with justification whether the following are true or false.

1. If H is a cyclic subgroup of $(\mathbb{R}, +)$ and $\mathbb{Z}$ is a subset of H, then H cannot contain any irrational number.

2. If A and B are two real 10×10 matrices such that all the entries of A are positive real numbers and all the entries of B are negative real numbers, then A cannot be similar to B.

3. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & x \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{0\} \\ \frac{1}{q}, & x = \frac{p}{q}, q \neq 0,\ gcd(p, q) = 1 \end{cases}$$

   is continuous at 0.

4. There exist linear transformations $S, T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $S + T$ is onto.

5. If $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$, $(n = 1, 2, \dots)$, then the sequence $\{s_n\}$ is convergent.

6. There exists only one ring homomorphism from $\mathbb{R}$ to $\mathbb{R}$.

7. For positive integers m and n, if $\phi(m) = \phi(mn)$ and $n > 1$, then $n = 2$ and m is odd, where ϕ is the Euler's phi-function, i.e., $\phi(m)$ is the number of positive integers less than m and relatively prime to m.

8. A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V^2(x)$$

where V is some differentiable function of x. Then at all times, either the difference between F and ma is zero, or the sum of kinetic and potential energy is zero.

9. The set of all spheres with centers on the z-axis is characterized by the partial differential equation

$$yp - xq = 0$$

10. The initial value problem

$$\frac{dy}{dx} = \sqrt{|y|}, \qquad\qquad y(0) = 0$$

has two solutions.

Unit II

(Each question of this unit carries 10 marks.)

11. Let G be the group of all rational numbers under addition and G' be the group of all positive rational numbers under multiplication. Determine all the group homomorphisms from G to G'.

12. Describe a one-one map from the set S of all ordered bases of $\mathbb{R}^2$ to $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$, the set of all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

13. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

is continuous but not differentiable at 0.

14. Let $n \geq 3$ be an integer such that $n^2 + 2^n$ is prime. Prove that $n \equiv 3 (mod\ 6)$.

15. Describe a homeomorphism from the unit disc $D = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \leq 1\}$ to the unit square $R = \{(x, y) \in \mathbb{R}^2 : -1 \leq x, y \leq 1\}$.

16. Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be two functions. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (u(x, y), v(x, y))$ and $g : \mathbb{C} \to \mathbb{C}$ be defined by $g(x + iy) = u(x, y) + iv(x, y)$. Determine with justification, which ones of the following are correct:

    (a) If f is differentiable on $\mathbb{R}^2$, then g is complex differentiable on $\mathbb{C}$.

    (b) If g is complex differentiable on $\mathbb{C}$, then f is differentiable on $\mathbb{R}^2$.

    (c) If f is continuous on $\mathbb{R}^2$, then g is continuous on $\mathbb{C}$.

    (d) If g is continuous on $\mathbb{C}$, then f is continuous on $\mathbb{R}^2$.

17. Let R be a ring having five elements. Determine with justification, which ones of the following are correct:

    (a) R is commutative.

    (b) R is a domain.

    (c) R is a field.

18. Reduce to cannonical form and find the general solution of

$$u_{xx} + 5u_{xy} + 6u_{xy} = 0$$

19. A particle moves in the xy plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is nonholonomic.

20. (a) If y_1, y_2 are twice continuously differentiable solutions of

$$y^{''} + a_1(t)y^{'} + a_0(t)y = 0$$

where a_1, a_0 are continuous on $I \subset \mathbb{R}$, then show that the Wronskian $W_{y_1y_2}$ satisfies

$$W^{'}_{y_1y_2} + a_1(t)W_{y_1y_2} = 0$$

(b) Find the Wronskian of two solutions of the equation

$$t^2y'' - t(t+2)y' + (t+2)y = 0; \qquad t > 0$$

21. (a) Explain why the combination of complementary function and particular integral gives the general solution of an ordinary differential equation.

(b) What is singular solution ?

(c) Describe how the solutions of ordinary differential equations are obtained near ordinary point, regular singular point and irregular singular point. Why solutions near those points are important ?

(d) Solve the equation

$$\frac{d^3y}{dx^3} - 5\frac{d^y}{dx^2} + 8\frac{dy}{dx} - 4y = e^{2x}.$$

22. (a) Show that if a particle describes a circular orbit under the influence of an attractive central force directed at a point on the circle, then the force varies as the inverse fifth power of the distance.

(b) Show that for the orbit described the total energy of the particle is zero

****