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# Kepler's Third Law, Dimensional Analysis and More

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## Abstract

*Kepler's third law states "The square of the period of a planet is proportional to the cube of its mean distance from the sun". Isaac Newton was the first person to derive this law using calculus, his laws of motion and the law of universal gravitation. We derive the Kepler's third law using dimensional analysis and plausibility arguments. Moreover, in the same way, we deduce the time period of closed orbits due to attractive linear forces.*

**Keywords: Kepler's third law, Newton's law of gravitation, Hooke's law,Dimensional analysis, Coulomb's law**

## A Historical Introduction

Aristotle (384-322 BC), pioneered occidental astronomy by constructing immovable celestial model of Universe. Claudius Ptolemy (2nd century AD) developed geometric model of geocentric universe. Taken together, these two models assume that planets, moons, sun are moving around earth; circular motion is the basic form of motion; beyond moon each and everything is unchanging. Contrasting heliocentric model of the universe was put forward much later by Nicolaus Copernicus (1473-1543 AD). With invention of better and better instruments, in 1500-1600 AD, varieties of models emerge within the extremes of geocentric and heliocentric viewpoints. One such model, theTychonic system (Brahe T. 1588), proposed that all planets excepting earth are revolving about sun, but sun in turn is rotating in circle about earth.

Tycho Brahe challenged the unchanging celestial, Aristotelian, perception by taking recourse to meticulous observation of the sky, night

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after night, through year after year. He discovered supernova SN1572[1](Brahe 1573) ripping through the conception of unchanging cosmos. Brahe, a Danish nobleman (wiki/Tycho_Brahe), devoted his career to science, was a naked eye astronomer, built a self-sufficient astronomy researchinstitute (West 2001) in the island of Hven in Øresund, did research in herbal medicine, published research papers based on celestial observations having been printed in his printing press, corresponded with his peers in and around Europe. He strived for accuracy of the order of minute of arc[2]. His data on Martian orbit were accurate upto two minutes of arc. He left Hven towards the end of sixteenth century, developed observatory at Prague. There in the year 1600, Kepler joined him as an assistant. He died almost accidentally, on 24th October, 1601, leaving the onus of publication of Martian Datato Johannes Kepler. Before death Tycho catalogued position of 1000 stars and also left behind an epitaph for himself, ''He lived like a sage and died like a fool''.

Kepler was a German and an avid Copernican (wiki/ Johannes_Kepler). The data collected by T. Brahe had its fruition in the hand of the mathematician Kepler. On fitting Tycho's all data to orbit, keeping the sun almost at the center, Kepler found that the Martian orbit is elliptical with the sun in one focus. All data excepting one were falling nicely on a circle. The exceptional data was at eight minutes of arc deviation from prevailing circular expectation. Kepler went ahead to generalize and put forward two laws in 1609 (Kepler 1609) and the third law in1619 (Kepler 1619). The trend-setting three laws are as follows:

First law: The orbit of each planet is an ellipse, with the sun at one of its foci.

Second law: The line joining the planet to the sun sweeps out equal areas in equal times.

Third law: The square of the period of a planet is proportional to the cube of its mean distance from the sun.

Kepler's three laws were descriptive not explanatory. One day in 1685 (Bate etal 1971), Edmund Halley, well known through Halley's Comet, with two of his contemporaries Christopher Wren and Robert Hooke, was toying with possible reasons for planetary motions. They speculated that a force like magnetism, falling off inversely with square of distance might not be behind the elliptical shape. Hooke volunteered to come with a proof but could not come up with one. Many months later, Halley was visiting Isaac

Newton at Cambridge. He casually posed a question to Newton, ‘If the sun pulled the planets with a force inversely proportional to the square of their distances, in what paths ought they to go?’’ Newton replied instantly, ‘’Why, in ellipse, of course...’’. He was referring to his work, done twenty years earlier. In 1666, during a long break at Cambridge due to plague outbreak, Newton, then twenty-three years old, conceived the laws of motion, the law of gravitation, and developed differential calculus. Moreover, he derived three laws of Kepler. Newton went ahead with the assumption of inverse square law force between two bodies, coupled with his laws of mechanics, to vindicate and supply the correct proportionality constant in Kepler’s third law. At the advice of Halley, he wrote and published his work in 1687(Newton 1687), “The Mathematical Principles of Natural Philosophy’’ or, simply ‘’Principia’’. The three laws of motion as enunciated in Principia are as follows:

First law: Everybody continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

Second law: The rate of change of momentum is proportional to the force impressed and is in the same direction as that force.

Third law: To every action there is always opposed an equal reaction.

Expressed mathematically, the second law appears as F=ma where, F is force impressed, m is mass and a is the produced acceleration respectively. Acceleration is change in velocity in unit time interval. Velocity is change in position in unit time interval.

Besides the three laws of motion in Principia, Newton described the law of universal gravitation which states as follows:

Any two bodies attract one another with a force proportional to the product of their masses and inversely proportional to the square of the distance between them. Expressed mathematically the law of gravitation reads as $F=\frac{Gm_1 m_2}{r^2}=\frac{k}{r^2}$.

The derivation of the third law of Kepler by Newton goes as follows (Chandrasekhar 1995): derive equation of ellipse under the gravitational force law, find the rate of temporal change of area vector, then integrate and put the area of ellipse equal to $\pi$ab where, a and b are semi-major and semiminor axes of the ellipse respectively. One deduces $a^3 = \frac{1}{4\pi^2}\frac{k}{\mu}T^2,$ where, μ is reduced mass[3] and T is orbital time period.

**Orbit Due to Force Law of Hooke**

Robert Hooke proposed the linear force law, which is the rule in case of a spring or, for elastic material, for small elongation. As in inverse square law force of attraction, for linear force also we get stable non-circular closed orbits, (Bertrand 1873, Goldstein 2002). Again following the steps as outlined in the previous paragraph, one arrives at for the closed orbits due to linear attractive force,

$$1 = \frac{1}{4\pi^2}\frac{k}{\mu}T^2.$$

Academic investigations on various fronts surrounding Kepler's third law went for centuries. It is going on unabated till today. We may get a feeling of recent researches, by looking into, (Dmitrasinovic etal 2015)for Kepler's third law for three body orbits, (Gorringeetal 1993) for elliptic orbits in presence of drag force, (Laskin 2013) for Kepler's third law in the context of deformed Newtonian Mechanics.

Can we avoid calculus and derive Kepler's third law in another way? If so, does the procedure work for another intervening force law, say, linear force law of Robert Hooke? We elaborate on one such heuristic approach as follows, after spending few lines here on dimensional analysis.

Mass, time and length are considered as fundamental quantities and rest other are considered as derived from these three. As a result, dimension of a derived quantity can be expressed as some power law function of dimensions of mass[M], time[T] and length[L]. Any equation in physics has to be dimensionally consistent i.e. both sides of an equation have to have the same dimension. From knowledge of dimension of one side knowing the dimension of other side is the essence of dimensional analysis. For details, see references (Halliday 2003,Bohren 2004). Apart from elementary aspect, dimensional analysis plays important role in advanced areas of physics like mechanical similarity in mechanics (Landau L. D.), renormalization group analysis (Goldenfeld N. 2005) etc.

**Kepler' Third Law**

Newton's law of gravitation is $F = \frac{Gm_1 m_2}{r^2} = \frac{k}{r^2}$ magnitudewise.

Simple dimensional analysis of the law of gravitation suggests, (Halliday 2003, Bohren 2004, Mungan 2009),

$[M][L][T]^{-2}$=[k]$[L]^{-2}$implying$[L]^3 = \frac{[k]}{[M]}[T]^2$ where, F = ma and [a]=[L].

Now, for two body problem, in the C.O.M frame, relevant mass is the reduced mass, μ; relevant length is semi-major axis length, a; relevant time is orbital time period, T, respectively. Hence,. What is the proportionality constant?

For closed orbit, it's two-dimensional motion i.e. one slice of spherical angle,$\theta$, which ranges from 0 to ð . Hence, the proportionality factor is $\frac{1}{\pi}$ multiplied by something else. To find that something else, we compare with electrostatic force rule in S.I. unit(Griffiths 1999, Jackson1996,Spavieri 2004), F=$\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$. In physical relationship what appears $\varepsilon_0$ not 4ð$\varepsilon_0$, say, in the Clausius-Mossotti formula.In physical relationship we divide $4\pi\varepsilon_0$ by 4π.

Analogue of 4ðin the case of Newton's law of gravitation is G, universal constant of gravitation. Hence, in the physical relation,

say, in the relation between semi-major axis length and time period,

we should divide G by 4ð. Hence is that something else. Therefore,

=

which is Kepler's third law, (Goldstein 2002).

Another way, we can deduce the The first mass is using one point out of 4ð solid angle through which gravitational field lines are emanating from the second mass, at any point of time. Hence, effective coupling istimes that of k.

Again, for Hooke's law,(Sommerfeld2003),F=kr, magnitude wise.

Simple dimensional analysis suggests

[M][L]=[k][L]implying 1=.

Now, for two body problem, in the C.O.M frame, relevant mass is the reduced mass, μ and relevant time is orbital time period, T. Hence,

1What is the proportionality constant?

Plausibility arguments along the same line as in the previous section suggests that the proportionality constant is .

Hence,1= .

This is the relation exhibiting length-scale independence of time-period of two body orbiting under mutual attractive force of Hooke's law type.

## Conclusion

We provide heuristic dimensional arguments for time periods, for closed orbits, in case of inverse square law and linear forces for two body motion, using few steps. The expressions are Kepler's third law and its analogue for harmonic oscillator potential. It will be interesting if arguments espoused here to fix the pre factor, can be used in other cases, say in three body problem.

This work may be useful for classroom teaching.

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## Notes

1 SN1572 is 7500 light-years away from earth.

2 For example, instrument for measurement of angle of one degree between two inclined planes costs Rs. 700, whereas instrument for measurement of angle of one minute between two inclined planes costs around Rs.30000. Precision comes at the cost of rise of price. Sometimes, one order of increase of accuracy costs one order of price rise.

3 Reduced mass = .

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