DEPARTMENT OF MATHEMATICS
NEHU, SHILLONG
M.Phil./Ph.D. Admission Test, Model 2019

Marks : 100
Time : Two hours

Attempt any 5 *questions from Unit I and any* 6 *from Unit II.*

Unit I

(Each question of this unit carries 8 *marks.)*

State with justification whether the following are true or false.

1. If x, y, z are elements of a group G such that $xyz = e$, then $yzx = e$, where e is the identity element of the group.
2. Let X be a compact metric space and Y be any metric space. If $f : X \to Y$ is a bijective, continuous map, then f^{-1} is also continuous.
3. If c_n is rational for all $n \in \mathbb{N}$ and Σc_n is convergent, then Σc_n is a rational number.
4. The ring $\mathbb{Z}_4$ is isomorphic to the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$.
5. There is an onto linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$.
6. If two $n \times n$ real matrices A and B have the same characteristic polynomial then they are similar.
7. There is a continuous and onto map from $(0, 1)$ to $\mathbb{Z}$.
8. There exists a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable only at the point 0.
9. If n solutions of an ordinary (linear nth order) differential equation are linearly dependent, their Wronskian vanishes.

10. The partial differential equation $xy^3u_{xx} - 2x^2y^2u_{xy} + yx^3u_{yy} - y^3u_x - x^3u_y = 0$ is elliptic.

Unit II

(Each question of this unit carries 10 marks.)

11. Show that a group of even order has at least one element which is its own inverse.

12. Let V be the vector space of real-valued functions on the interval $[0, 1]$ over $\mathbb{R}$. Show that the functions $x^3, \sin x$ and $\cos x$ are linearly independent elements of V.

13. Determine with proper justification which of the following subsets of $\mathbb{R}^3$ can be written as a cartesian product of subsets of $\mathbb{R}$:

    (a) $\{(x_1, x_2, x_3) : x_2 = x_3\}$.

    (b) $\{(x_1, x_2, x_3) : x_i \geq i, \forall i = 1, 2, 3\}$.

14. Determine all the connected subsets of $\mathbb{Z}$.

15. Determine whether the direct product of two infinite cyclic groups is again infinite cyclic.

16. Show that the map $\theta : \mathbb{Z}[x] \to \mathbb{R}$ given by $\theta(f(x)) := f(\sqrt{2})$ is a ring homomorphism. Find out the kernel of θ.

17. Find the radius of convergence of the power series $\sum \frac{1}{n!} z^n$

18. Let L, R and C be positive constants and consider the equation $Ly'' + Ry' + \frac{1}{C}y = 0$. Compute all solutions for the cases $\frac{R^2}{L^2} - \frac{4}{LC} > 0$ and $\frac{R^2}{L^2} - \frac{4}{LC} = 0$.

19. What is the remainder when $42^{61} + 18^{31} + 15$ is divided by 31?

20. Is it true that for any $n \geq 1$ the integer $32^n + 1$ is composite? Justify your answer.

21. Find a linear transformation on $\mathbb{R}^2$ which takes the line $L_1 = \{(x, 3x) : x \in \mathbb{R}\}$ to the line $L_2 = \{(2x, 5x) : x \in \mathbb{R}\}$.

22. If for some indexing set I, $\{A_\alpha : \alpha \in I\}$ is a family of subsets of a topological space X, then prove that $\overline{\cup A_\alpha} \supseteq \cup \overline{A_\alpha}$. Give an example to show that equality may fail.

****